

A PROBLEM OF KUSNER ON EQUILATERAL SETS

KONRAD J. SWANEPOEL

ABSTRACT. R. B. Kusner [R. Guy, Amer. Math. Monthly **90** (1983), 196–199] asked whether a set of vectors in \mathbb{R}^d such that the ℓ_p distance between any pair is 1, has cardinality at most $d + 1$. We show that this is true for $p = 4$ and any $d \geq 1$, and false for all $1 < p < 2$ with d sufficiently large, depending on p . More generally we show that the maximum cardinality is at most $(2\lceil p/4 \rceil - 1)d + 1$ if p is an even integer, and at least $(1 + \varepsilon_p)d$ if $1 < p < 2$, where $\varepsilon_p > 0$ depends on p .

1. INTRODUCTION

Let $1 < p < \infty$ and $d \geq 1$. By ℓ_p^d we denote \mathbb{R}^d endowed with the ℓ_p -norm

$$\|x\|_p = \|(x_1, \dots, x_d)\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{1/p}.$$

The *unit ball* and *unit sphere* (*unit circle* if $d = 2$) of ℓ_p^d are the sets $\{x : \|x\|_p \leq 1\}$ and $\{x : \|x\|_p = 1\}$, respectively. Note that we do not consider the cases $p = 1, \infty$ in this paper. A set $S \subset \ell_p^d$ is λ -*equilateral* ($\lambda > 0$) if $\|x - y\|_p = \lambda$ for all distinct $x, y \in S$, and *equilateral* if S is λ -equilateral for some $\lambda > 0$. The maximum number of elements in an equilateral set in ℓ_p^d is denoted by $e(\ell_p^d)$. It is well-known that $e(\ell_2^d) = d + 1$. The standard basis vectors of \mathbb{R}^d together with some multiple of $(1, 1, \dots, 1)$ demonstrates that $e(\ell_p^d) \geq d + 1$ for all $1 < p < \infty$. A result of Petty [8] gives as a special case that $e(\ell_p^d) < 2^d$ for $d \geq 2$. It is also well-known that $e(\ell_p^2) = 3$ (see e.g [6, Section 5]). Kusner [4] asked whether $e(\ell_p^d) = d + 1$ for all $d \geq 2$ and $1 < p < \infty$. This problem has recently been studied by Smyth [9] and Alon and Pudlák [1]. Smyth showed $e(\ell_p^d) < c p d^{(p+1)/(p-1)}$ for some $c > 0$, and also $e(\ell_p^d) = d + 1$ for $2 - \alpha_d < p < 2 + \alpha_d$ where $\alpha_d = O(1/(d \log d))$ (this second statement also follows from a more general result of Brass [2] and Dekster [3]). The general upper bound was improved by Alon and Pudlák to $e(\ell_p^d) < c_p d^{(2p+2)/(2p-1)}$ for some $c_p > 0$ depending on p . For p an even integer, Galvin (see [9]) showed $e(\ell_p^d) \leq 1 + (p - 1)d$, while for p an odd integer Alon and Pudlák showed $e(\ell_p^d) \leq c_p d \log d$ for some $c_p > 0$.

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First of all we improve Galvin's result as follows.

Theorem 1. *For p an even integer and $d \geq 1$ we have*

$$e(\ell_p^d) \leq \begin{cases} (\frac{p}{2} - 1)d + 1 & \text{if } p \equiv 0 \pmod{4}, \\ \frac{p}{2}d + 1 & \text{if } p \equiv 2 \pmod{4}. \end{cases}$$

In particular, $e(\ell_4^d) = d + 1$.

By a compactness argument we thus have that $e(\ell_p^d) = d + 1$ for p in a small interval around 4, the size of the interval depending on d . We have no information on the size of this interval, nor on whether $e(\ell_p^d) = d + 1$ for any other values of $p > 2$. The proof of Theorem 1 in Section 2 uses a linear algebra method (see [5, Part III] for an exposition).

Secondly we show that $e(\ell_p^d) > d + 1$ holds for all $1 < p < 2$ if d is sufficiently large.

Theorem 2. *For any $1 < p < 2$ and $d \geq 1$, let*

$$k = \left\lceil \frac{\log(1 - 2^{p-2})^{-1}}{\log 2} \right\rceil - 1.$$

Then

$$e(\ell_p^d) \geq \left\lfloor \frac{2^{k+1}}{2^{k+1} - 1} d \right\rfloor.$$

In particular, if $d \geq 2^{k+2} - 2$ then $e(\ell_p^d) > d + 1$.

For example, if $1 < p \leq \frac{\log 3}{\log 2}$, then $e(\ell_p^d) \geq \lfloor 4d/3 \rfloor$ and $e(\ell_p^6) \geq 8$. For p close to 2 the theorem implies that $e(\ell_p^d) > d + 1$ if $p < 2 - \Omega(1/d)$. Thus we have reached the above-mentioned bound of Smyth except for a $\log d$ factor. Theorem 2 is proved in Section 3 by constructing explicit examples based on Hadamard matrices.

The smallest dimension for which Theorem 2 gives an example of $e(\ell_p^d) > d + 1$ is $d = 6$. With a slightly modified construction we also give examples for $d = 4$. However, we have no examples for $d = 3$ or $d = 5$.

Theorem 3. *For any $1 < p \leq \frac{\log 5/2}{\log 2}$ we have $e(\ell_p^4) \geq 6$.*

The proof is also in Section 3.

2. UPPER BOUNDS FOR p AN EVEN INTEGER

Let S be a 1-equilateral set in ℓ_p^d where p is an even integer. For each $a \in S$, let $P_a(x) = P_a(x_1, \dots, x_d)$ be the following polynomial:

$$\begin{aligned} P_a(x) &:= -1 + \|x - a\|_p^p \\ &= -1 + \|a\|_p^p + \sum_{i=1}^d x_i^p + \sum_{i=1}^d \sum_{m=1}^{p-1} \binom{p}{m} (-a_i)^{p-m} x_i^m. \end{aligned} \quad (1)$$

Thus each P_a is in the linear span of

$$\left\{1, \sum_{i=1}^d x_i^p\right\} \cup \{x_i^m : 1 \leq m \leq p-1; 1 \leq i \leq d\},$$

which is a subspace of dimension $(p-1)d + 2$ of the vector space of real polynomials in the variables x_1, \dots, x_d . Since $P_a(a) = -1$ for all $a \in S$ and $P_a(b) = 0$ for all distinct $a, b \in S$, we have that $\{P_a : a \in S\}$ is linearly independent. Thus we already have $|S| \leq (p-1)d + 2$. We now show that the larger set

$$\mathcal{P} := \{P_a : a \in S\} \cup \{1\} \cup \{x_i^m : 1 \leq i \leq d; 1 \leq m \leq k\}$$

is still linearly independent, where $k = p/2$ if $p \equiv 0 \pmod{4}$ and $k = p/2 - 1$ otherwise. This will give $|S| + 1 + kd \leq (p-1)d + 2$, proving Theorem 1.

We only consider the case $p \equiv 0 \pmod{4}$, the other case being similar. Let λ, λ_a ($a \in S$), $\lambda_{i,m}$ ($1 \leq i \leq d; 1 \leq m \leq p/2$) be real numbers satisfying

$$\lambda 1 + \sum_{a \in S} \lambda_a P_a(x) + \sum_{i=1}^d \sum_{m=1}^{p/2} \lambda_{i,m} x_i^m \equiv 0. \quad (2)$$

If we substitute (1) into (2) we obtain

$$\begin{aligned} \lambda 1 + \sum_{a \in S} \lambda_a (-1 + \|a\|_p^p) + \sum_{i=1}^d \left(\sum_{a \in S} \lambda_a \right) x_i^p \\ + \sum_{i=1}^d \sum_{m=1}^{p-1} \sum_{a \in S} \lambda_a \binom{p}{m} (-a_i)^{p-m} x_i^m + \sum_{i=1}^d \sum_{m=1}^{p/2} \lambda_{i,m} x_i^m \equiv 0. \end{aligned} \quad (3)$$

Thus the coefficients of this polynomial are all 0, giving

$$\lambda + \sum_{a \in S} \lambda_a (-1 + \|a\|_p^p) = 0, \quad (4)$$

$$\sum_{a \in S} \lambda_a = 0, \quad (5)$$

$$\lambda_{i,m} + \sum_{a \in S} \lambda_a \binom{p}{m} (-a_i)^{p-m} = 0 \quad \forall m = 1, \dots, p/2; i = 1, \dots, d, \quad (6)$$

$$\sum_{a \in S} \lambda_a a_i^m = 0 \quad \forall m = 1, \dots, p/2 - 1. \quad (7)$$

Substitute $x = b \in S$ into (2):

$$-\lambda_b + \lambda + \sum_{i=1}^d \sum_{m=1}^{p/2} \lambda_{i,m} b_i^m = 0 \quad \forall b \in S. \quad (8)$$

Multiply (8) by $-\lambda_b$ and sum over all $b \in S$:

$$\sum_{b \in S} \lambda_b^2 - \lambda \sum_{b \in S} \lambda_b - \sum_{i=1}^d \sum_{m=1}^{p/2} \lambda_{i,m} \sum_{b \in S} \lambda_b b_i^m = 0.$$

By (5) and (7) this simplifies to

$$\sum_{b \in S} \lambda_b^2 - \sum_{i=1}^d \lambda_{i,p/2} \sum_{b \in S} \lambda_b b_i^{p/2} = 0,$$

which by (6) simplifies to

$$\sum_{b \in S} \lambda_b^2 + \sum_{i=1}^d \binom{p}{m} \left(\sum_{a \in S} \lambda_a a_i^{p/2} \right)^2 = 0.$$

Since the left-hand side is a sum of squares, $\lambda_b = 0$ for all $b \in S$. By (4) we then have $\lambda = 0$, and by (6) $\lambda_{i,m} = 0$ for all m and i . Thus the set \mathcal{P} is linearly independent, finishing the proof. \square

3. LOWER BOUNDS FOR $1 < p < 2$

According to the following proposition, if we can find a $2^{1/p}$ -equilateral set of $k+1$ points on the unit sphere of ℓ_p^k , we can construct equilateral sets in ℓ_p^d of more than $d+1$ points if d is sufficiently large. The construction is similar to the Lenz construction in combinatorial geometry (see [7, pp. 148, 159, 194]).

Proposition 1. *Let $1 < p < \infty$, $k, d \geq 1$. If ℓ_p^k has a $2^{1/p}$ -equilateral set of cardinality $k+1$ on the unit sphere, then ℓ_p^d has a $2^{1/p}$ -equilateral set of cardinality $\lfloor (1 + \frac{1}{k})d \rfloor$ on the unit sphere.*

Proof. Let $m = \lfloor d/k \rfloor$ and $r = d - km$. Let $\ell_p^d = \overbrace{\ell_p^k \oplus \cdots \oplus \ell_p^k}^{m \text{ times}} \oplus \ell_p^r$. Let the equilateral set in ℓ_p^k be $S = \{v_1, \dots, v_{k+1}\}$. Let S_i be the copy of S in the i 'th copy of ℓ_p^k in ℓ_p^d , $i = 1, \dots, m$, and let S_0 be the copy of the standard unit vectors e_1, \dots, e_r in the copy of ℓ_p^r , which is also a $2^{1/p}$ -equilateral set of unit vectors. Clearly the distance between a vector in S_i and a vector in S_j is $2^{1/p}$ for distinct i, j , since both are unit vectors. Thus $S_0 \cup \cdots \cup S_m$ is the required set, since it has cardinality $m(k+1) + r = d + m = d + \lfloor d/k \rfloor$. \square

Before we construct the required $2^{1/p}$ -equilateral sets, we need a technical two-dimensional result.

Lemma 1. *Let $1 < p < 2$. For each $\lambda \in [2^{1-1/p}, 2^{1/p}]$ there exist unit vectors $u, v \in \ell_p^2$ such that $\|u + v\|_p = \|u - v\|_p = \lambda$.*

Geometrically the lemma says that there exists a quadrilateral inscribed in the unit circle of ℓ_p^2 with all four sides of length λ , for any $\lambda \in [2^{1-1/p}, 2^{1/p}]$. This is easily seen for $\lambda = 2^{1/p}$ ($u = (1, 0)$ and $v = (0, 1)$) and for $\lambda = 2^{1-1/p}$

($u = ((1/2)^{1/p}, (1/2)^{1/p})$ and $v = (-(1/2)^{1/p}, (1/2)^{1/p})$). The inbetween values are then covered by a continuity argument. We omit the details.

We also need Hadamard matrices. Recall that a Hadamard matrix of order k is a $k \times k$ matrix H with all entries ± 1 , satisfying $HH^T = kI$. It is well-known that if a Hadamard matrix of order k exists, then $k = 1, 2$ or k is divisible by 4 [10], and not known whether the converse holds. However, we only need the fact that Hadamard matrices of order 2^n exist for all $n \geq 0$, as shown by the well-known inductive construction

$$H_0 = [1], \quad H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}, n \geq 0.$$

Proposition 2. *Suppose that there exists a Hadamard matrix of order $k \geq 2$. Then for any*

$$p \in \left[2 + \frac{\log(1 - k^{-1})}{\log 2}, 2 + \frac{\log(1 - (2k)^{-1})}{\log 2} \right], \quad p \neq 1,$$

there exists a $2^{1/p}$ -equilateral set of cardinality $2k$ on the unit sphere of ℓ_p^{2k-1} .

Proof. We may assume without loss of generality that the $k \times k$ Hadamard matrix is normalized such that its first column contains only $+1$'s. Delete the first column and let the k resulting rows be $w_1, \dots, w_k \in \{\pm 1\}^{k-1}$. We have that any two distinct w_i and w_j differ in exactly $k/2$ coordinates. Let

$$\lambda = 2 \left(\frac{(3 - 2^{p-1})k - 2}{2(k-1)} \right)^{1/p}. \quad (9)$$

The given bounds on p ensure that $2^{1-1/p} \leq \lambda \leq 2^{1/p}$. We now take $u, v \in \ell_p^2$ from Lemma 1, and let

$$u_i = (\mu, w_i \otimes u), \quad v_i = (-\mu, w_i \otimes v) \in \ell_p^{2k-1}, \quad i = 1, \dots, k,$$

where

$$\mu = ((2^{p-2} - 1)k + 1)^{1/p} \quad (10)$$

and the *Kronecker product* $a \otimes b$ for any vectors $a = (a_1, \dots, a_m) \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ is defined as $(a_1b, a_2b, \dots, a_mb) \in \mathbb{R}^{mn}$. The given lower bound on p ensures that μ is well-defined. Then for any distinct i, j we have $\|u_i - u_j\|_p^p = (k/2)\|2u\|_p^p = 2^{p-1}k$ and similarly $\|v_i - v_j\|_p^p = 2^{p-1}k$. Also for any i, j ,

$$\begin{aligned} \|u_i - v_j\|_p^p &= (2\mu)^p + \sum_{m=1}^{k-1} \|u + \varepsilon_m v\|_p^p \quad \text{for some } (\varepsilon_m) \in \{\pm 1\}^{k-1} \\ &= (2\mu)^p + (k-1)\lambda^p \quad \text{by Lemma 1} \\ &= 2^{p-1}k \quad \text{by (9) and (10).} \end{aligned}$$

Thus $S := \{u_1, \dots, u_k, v_1, \dots, v_k\}$ is equilateral. Also,

$$\|u_i\|_p^p = \mu^p + (k-1)\|x\|_p^p = \mu^p + k - 1 = 2^{p-2}k,$$

by (10), and similarly, $\|v_i\|_p^p = 2^{p-2}k$. Thus if we scale S by $(2^{p-2}k)^{-1/p}$, we obtain a $2^{1/p}$ -equilateral set of unit vectors of cardinality $2k$. \square

Note that the two smallest dimensions d for which the above proposition ensures a $2^{1/p}$ -equilateral set of unit vectors of size $d + 1$ are $d = 3$ (with $1 < p \leq \log 3 / \log 2$) and $d = 7$ (with $\log 3 / \log 2 \leq p \leq \log(7/2) / \log 2$). It is not difficult to see that such a set does not exist for $d = 2$ and any $1 < p < \infty$, and also not for $p = 2$ and any d . We do not know whether such sets exist if $d = 4, 5, 6$. It is doubtful that they exist for $p > 2$.

Proof of Theorem 2. Note that the given value of k ensures that

$$2 + \frac{\log(1 - 2^{-k})}{\log 2} < p \leq 2 + \frac{\log(1 - 2^{-k-1})}{\log 2},$$

and thus Proposition 2 applied to a Hadamard matrix of order 2^k gives a $2^{1/p}$ -equilateral set of cardinality 2^{k+1} on the unit sphere of $\ell_p^{2^{k+1}-1}$. Then by Proposition 1 ℓ_p^d has a $2^{1/p}$ -equilateral set of size $\lfloor (1 + (2^{k+1} - 1)^{-1})d \rfloor$. \square

Proof of Theorem 3. Let $\lambda = 2(3 - 2^p)^{1/p}$ and take $u, v \in \ell_p^2$ from Lemma 1. (Since $1 < p \leq \log(5/2) / \log 2$ we have $2^{1-1/p} \leq \lambda \leq 2^{1/p}$.) Let $\mu = (2^p - 2)^{1/p}$. Then a simple calculation shows that

$$S = \{(\mu, \pm u, 0), (-\mu, \pm v, 0), (0, o, \pm 1)\} \subset \ell_p^4$$

is equilateral. \square

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DEPARTMENT OF MATHEMATICS, APPLIED MATHEMATICS AND ASTRONOMY, UNIVERSITY OF SOUTH AFRICA, PO BOX 392, PRETORIA 0003, SOUTH AFRICA
E-mail address: swanekj@unisa.ac.za